Using the Differential Equations Special Functions Library (despecfunc.tns)

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1   Purpose

The functions in the differential equations special function library,`despecfunc.tns`, implement several of the special functions for solving complicated differential equations. The functions in the library are Bessel functions of the first and second kind, Legendre functions of the first and second kind, and the Gamma and Log Gamma functions. These are the special functions usually encountered in an introductory or intermediate course in differential equations.

The library can be installed and used with TI-Nspire CX CAS, TI-Nspire CAS Student Software, TI-Nspire CAS Teacher Software, and TI-Nspire CAS App for the iPad.

`despecfunc.tns` is free software: you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation, either version 3 of the License, or any later version. Visit `https://www.gnu.org/licenses/gpl.htm` to view the License.

2   Restrictions/Limitations

1. The expressions for the Legendre polynomials returned from the two Legendre functions are defined with the symbolic variable **x**. The variable must not have a value assigned to it before executing the functions, since TI-Nspire will replace the variable with its assigned value when the expression is returned from the functions. Use `delVar x` before executing the functions to avoid this side-effect.

2. The library functions require the CAS version of TI-Nspire.

3   Library History

Author: Forest W. Arnold
Date Created: 25 May 2020
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Version: 1.1

4   Function Descriptions

4.1   Bessel's Equation

Bessel's equation of order v is the differential equation

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

The Bessel functions, $J_n(x)$ and $Y_n(x)$ are solutions of Bessel's equation. $J_n(x)$ is called the *Bessel function of the the first kind* and $Y_n(x)$ is called the *Bessel function of the*

second kind. $J_n(x)$ and $Y_n(x)$ are linearly independent solutions of Bessel's equation, and the general solution on the interval $(0, \infty)$ is the linear combination

$$c_1 J_n(x) + c_2 Y_n(x)$$

The library functions `besseljn(n,x)` and `besselyn(n,x)` return the values of $J_n(x)$ and $Y_n(x)$ as floating-point numbers. The 12 decimal digit return values for Bessel's equation of order $\nu = 3$ and $x = 10$ are

jn3:=despecfunc\besseljn$(3,10)$	0.058379379305
yn3:=despecfunc\besselyn$(3,10)$	-0.251362657184

The library function `besselvalues(n,x,kind)` returns the values of $J_n(x)$ or $Y_n(x)$ and the first and second derivatives of the function at a value of x. The values are returned in a list: {`function value,first derivative,second derivative`}. Examples are

j0vals:=despecfunc\besselvalues$(0,5,1)$ $\quad \{-0.177596771314, 0.327579137591, 0.112080943796\}$

y0vals:=despecfunc\besselvalues$(0,5,2)$

$$\{-0.308517625249, -0.147863143391, 0.338090253927\}$$

4.1.1 Graphs of the Bessel Functions

Graphs of the first five Bessel functions of the first kind are shown in Figure 1 and graphs of the first five Bessel functions of the second kind are shown in Figure 2.

Figure 7 and Figure 8 in Example 2 below show graphs of the Bessel functions and their first and second derivatives for $J_3(x)$ and $Y_3(x)$ for $x = 4.5$.

4.1.2 Code for the Bessel Functions

The implementation for the TI-Nspire version of the code is based on the optimized high-precision C++ source code in the file *specialfunctions.cpp*. The code is part of the GPL 2+ version of Alglib 3.16.0, available at `https://www.alglib.net/download.php`.

The implementation of `besselvalues()` is based on extensions of formulas for the derivatives of the Bessel functions from an article titled "On the positive zeros of the second derivative of Bessel Functions", by E.K. Ifantis, C.G. Kokologiannaki, and C.B. Kouris, published in "Journal of Computational and Applied Mathematics 34 (1991) 27 -31".

4.2 Legendre's Equation

Legendre's equation of order n is the differential equation

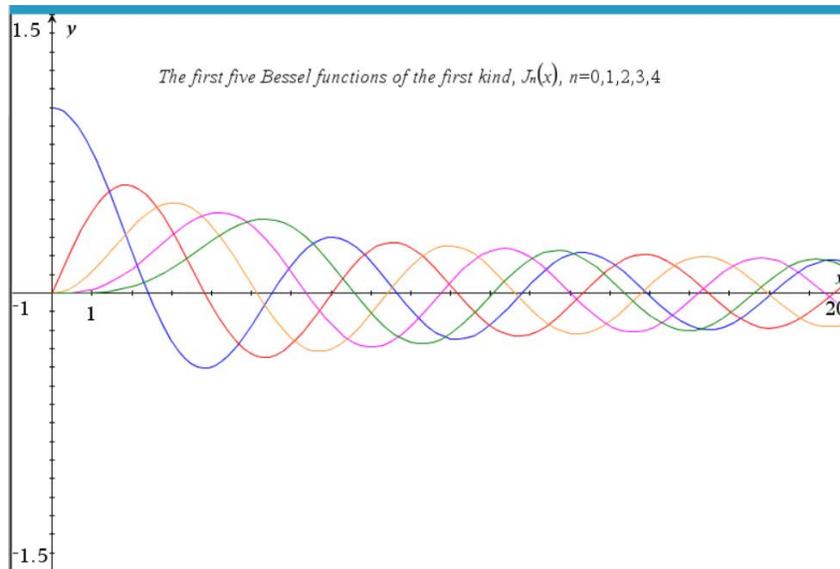$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$
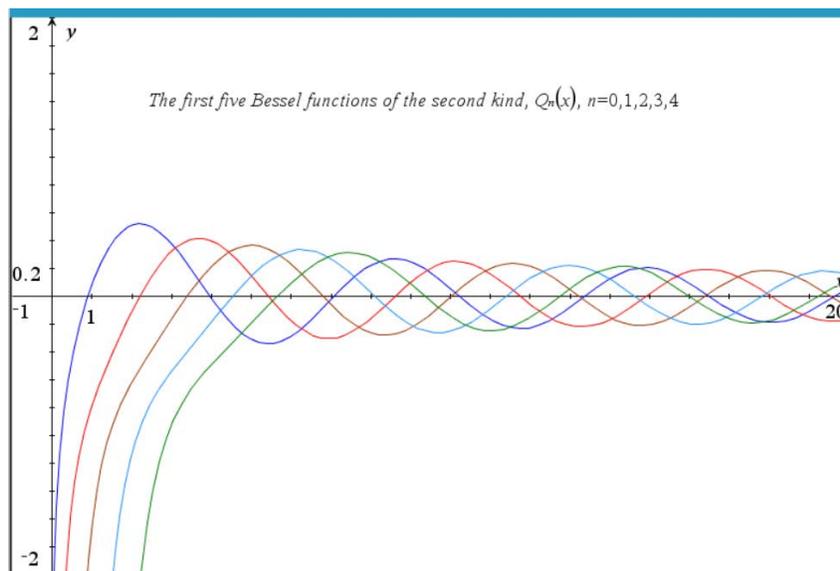
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Figure 1: Bessel Functions of the First Kind

Figure 2: Bessel Functions of the Second Kind

The Legendre functions, $P_n(x)$ and $Q_n(x)$ are solutions of Legendre's equation. $P_n(x)$ is called the *Legendre function of the the first kind* and $Q_n(x)$ is called the *Legendre function of the second kind*. $P_n(x)$ and $Q_n(x)$ are linearly independent solutions of

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Legendre's equation. $P_n(x)$ is bounded and defined on the interval $[-1,1]$. $Q_n(x)$ is undefined at -1 and 1 and defined on the open interval $(-1,1)$. The general solution on the interval $(-1,1)$ is the linear combination

$$c_1 P_n(x) + c_2 Q_n(x)$$

The library functions `legendrepn()` and `legendreqn()` return the solutions $P_n(x)$ and $Q_n(x)$. The value returned for $P_n(x)$ is a polynomial. The value returned for $Q_n(x)$ is a logarithmic function. Both values are expressions containing the symbolic variable x, as shown by the independent solutions to the Legendre equation $(1-x^2)y'' - 2xy' + 12y = 0$ where $n = 3$:

DelVar x	*Done*
$p3x:=despecfunc\backslash legendrepn(3)$	$\dfrac{x \cdot \left(5 \cdot x^2 - 3\right)}{2}$
$q3x:=despecfunc\backslash legendreqn(3)$	$\dfrac{3 \cdot x \cdot \left(5 \cdot x^2 - 3\right) \cdot \ln\left|\dfrac{-(x+1)}{x-1}\right| - 2 \cdot \left(15 \cdot x^2 - 4\right)}{12}$

4.2.1 Graphs of the Legendre Functions

Graphs of the first five Legendre functions of the first kind are shown in Figure 3:
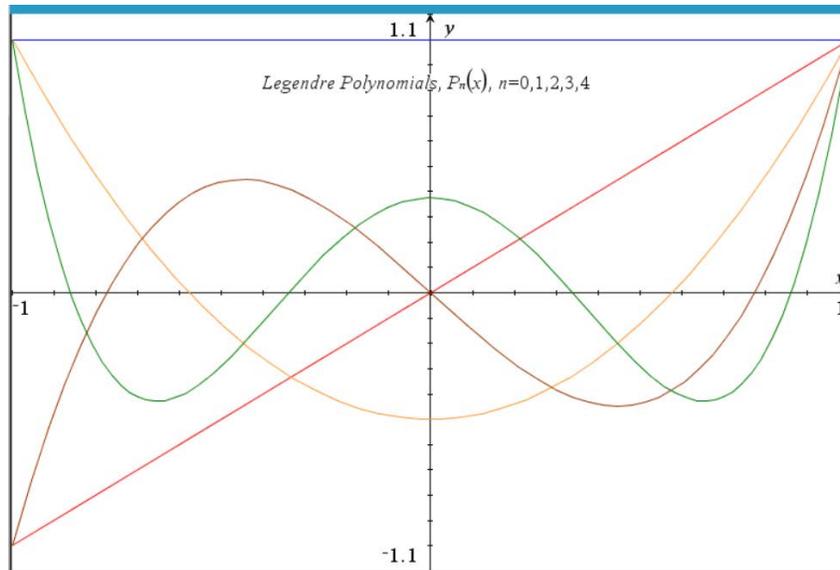

Figure 3: Legendre Functions of the First Kind

Graphs of the first five Legendre functions of the second kind are shown in Figure 4:
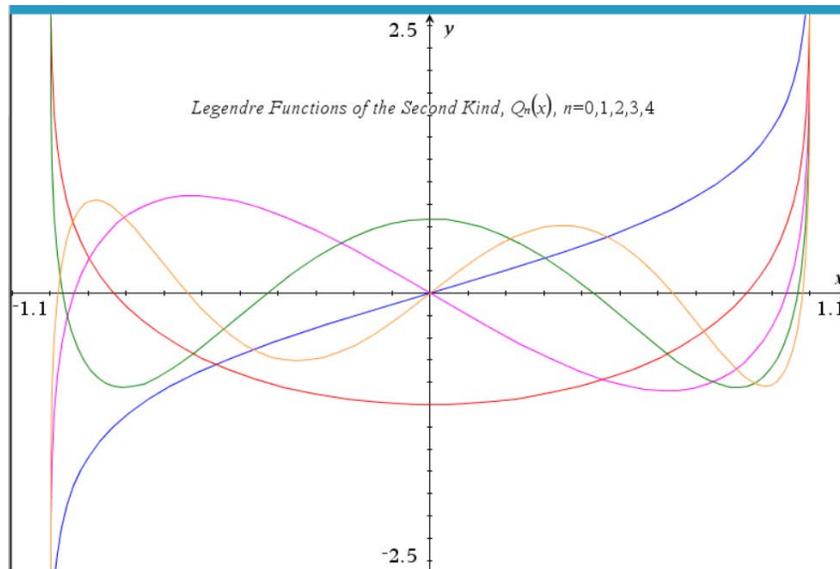
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Figure 4: Legendre Functions of the Second Kind

4.2.2 Code for the Legendre Functions

The implementation of the Legendre function of the first kind is based on Bonnet's recursion formula:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

with $P_0(x) = 1$ and $P_1(x) = x$.

The implementation of the Legendre function of the second kind is based on the recursion formula

$$Q_n(x) = \frac{1}{n}[(2n-1)Q_{n-1}(x) - (n-1)Q_{n-2}(x)]$$

with $Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ and $Q_1(x) = xQ_0(x) - 1$.

4.3 Gamma Function

The Gamma function, $\Gamma(x)$, generalizes the factorial function, $n!$, to the complex numbers. This function was first introduced by Leonhard Euler. It has been extensively studied by many famous mathematicians for over 300 years. It appears in many areas of advanced and applied mathematics such as asymptotic series, bessel functions, hypergeometric series, Riemann-Zeta functions, mathematical statistics, *et.al.* Detailed information about the Gamma function is found at the link

https://en.wikipedia.org/wiki/Gamma_function

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The library function, `gamma(x)`, returns the floating point value of the Gamma function for the input variable, x. Some values returned by the function are

	A x	B gamma
=		=despecfunc\gamma('x)
1	-3/2	2.36327180121
2	-1/2	-3.54490770181
3	0	#UNDEF
4	1/2	1.77245385091
5	1	1.
6	3/2	0.886226925453
7	2	1.
8	5/2	1.32934038818
9	3	2.
10	7/2	3.32335097045
11	4	6.

4.3.1   Graph of the Gamma Function

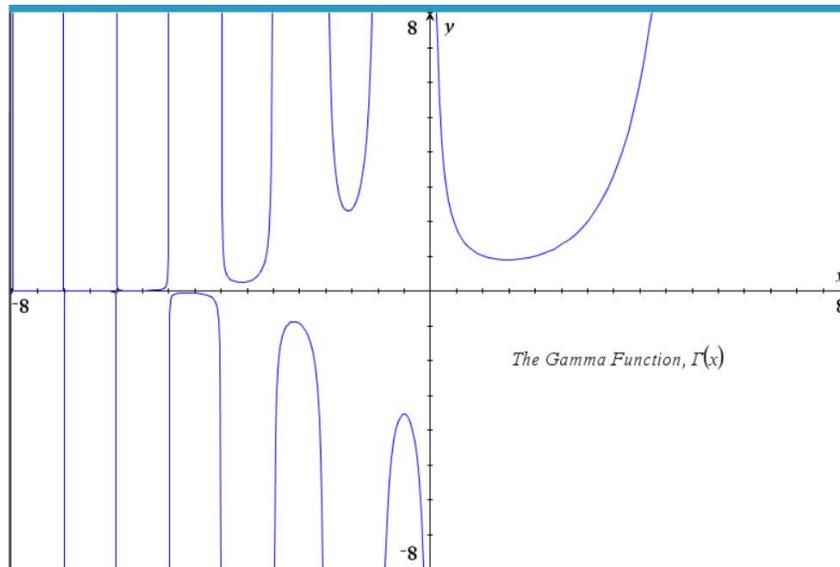A graph of the Gamma function is shown in Figure 5:

Figure 5: The Gamma Function, $\Gamma(x)$

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4.3.2 Code for the Gamma Function

The code implementing the library function, `gamma(x)`, is based on a javascript implementation of the Lanczos approximation to the Gamma function. The javascript source is available at the link

https://rosettacode.org/wiki/Gamma_function#JavaScript.

Information about the Lanczos approximation method, including an algorithm for the method, can be found at the link

https://en.wikipedia.org/wiki/Lanczos_approximation.

4.4 Log-Gamma Function

The Log-Gamma function, $\ln\Gamma(x)$, returns the value of the natural logarithm of the Gamma function for the value of the variable x. Since values of the Gamma function quickly become large with increasing values of x, the Log-Gamma function is frequently used as an alternative to the Gamma function. Instead of multiplying and dividing large values from the Gamma function, the same results can be achieved by adding and subtracting the smaller values returned by the Log-Gamma function. Additionally, values returned from the Log-Gamma function are easily transformed to Gamma function values.

The library function named `lngamma(x)` implements the Log-Gamma function. Some values returned by the function are

	A x	B lngamma
=	=seq(n,n,-3/2,4,1/2)	=approx(despecfunc\lngamma('x))
1	-3/2	1.5532061309105−6.28318530...
2	-1	1.8381845649507−4.71238898...
3	-1/2	2.0184349201139−3.14159265...
4	0	#UNDEF
5	1/2	0.512589
6	1	-0.000307
7	3/2	-0.120794
8	2	-0.000001
9	5/2	0.284683
10	3	0.693147
11	7/2	1.20097
12	4	1.79176

Notice that the values returned for negative values of x are complex numbers.

4.4.1 Graph of the Log-Gamma Function

A graph of the Log-Gamma function, including the real parts of the complex values for the negative x values is shown in Figure 6:
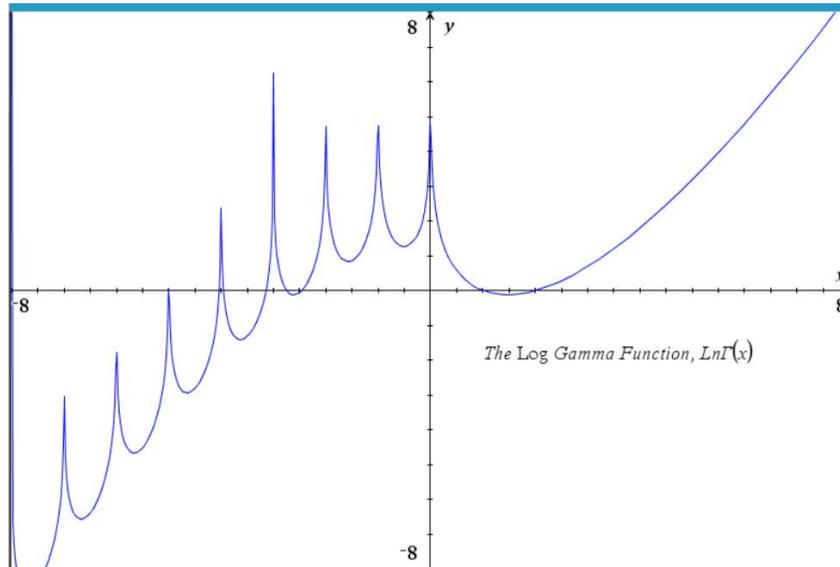

Figure 6: The Log-Gamma Function, $\ln\Gamma(x)$

4.4.2 Code for the Log-Gamma Function

The code for `lngamma(x)` is based on C code for a Log-Gamma function in the book *More C Tools for Scientists and Engineers* by Louis Baker and published by McGraw-Hill, Inc., in 1991.

5 Function Usage

The following paragraphs describe each of the public functions in the library.

5.1 help()

Displays usage information for the functions in the library.

Select the help function in the library pane to add it to a calculator page, then execute the function:

```
de_specfunc\help()
```

Help for the de_specfunc library.
==

Description:
A few Special Functions that are usually introduced in a first course in
differential equations.

Author: Forest W. Arnold
Version: 1.0
Date: May 2020
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it under the terms of the GNU General Public License as published by
the Free Software Foundation, either version 3 of the License, or
any later version.

Scroll to display information about a library function:

Public Functions:

1. besseljn(n,x)
Purpose:
    Calculate Jn(x), value of bessel function of the first kind of integer
    order n for x.
Input Arguments:
    n – the integer for the order: 0,1,2,...,n.
    x – the independent variable.
Returns:
    The value of the bessel function of the first kind of integer
    order n for x.
————
2. besselyn(n,x)
Purpose:
    Calculate Yn(x), value of bessel function of the second kind of integer

5.2   besseljn(n,x)

Calculate and return the value $J_n(x)$, of the bessel function of the first kind of order n
for the independent variable x.

Input arguments:

n - the order for the function: $0, 1, 2, ..., n$.

x - the value of the independent variable.

Returns:

The value, $J_n(x)$, corresponding to the value of the independent variable.

Note: The value of n must be an integer. Negative and/or non-integer values are co-erced to integer values.

5.3   besselyn(n,x)

Calculate and return the value $Y_n(x)$, of the bessel function of the second kind of order n for the independent variable x.

Input arguments:

n - the order for the function: $0, 1, 2, ..., n$.

x - the value of the independent variable.

Returns:

The value, $Y_n(x)$, corresponding to the value of the independent variable.

Note: The value of n must be an integer. Negative and/or non-integer values are co-erced to integer values.

5.4   besselvalues(n,x,kind)

Calculate and return the value $J_n(x)$ or $Y_n(x)$, of the bessel function of the first or second kind of order n for the independent variable x along with the values of the first and second derivatives of the function at x.

Input arguments:

n - the order for the function: $0, 1, 2, ..., n$.

x - the value of the independent variable.

kind - integer value 1 or 2: $1=J_n(x)$, $2=Y_n(x)$.

Returns:

A list containing the values, either $\{J_n(x), J'_n(x), J''_n(x)\}$ or $\{Y_n(x), Y'_n(x), Y''_n(x)\}$, depending on whether `kind` is 1 or 2.

Notes: 1. The value of n must be an integer. Negative and/or non-integer values are

coerced to positive integer values.

2. The value of `kind` must be 1 or 2. Any other value is changed to 1.

5.5   legendrepn(n)

Construct and return $P_n(x)$, the Legendre polynomial of the first kind.

Input arguments:

>n - designates which polynomial to construct: $0, 1, \cdots, n$ specifies
>$P_0(x), P_1(x), \cdots, P_n(x)$ respectively.

Returns:

A polynomial in the symbolic variable x; *e.g.,* $\frac{3x^2-1}{2}$.

Notes:

>n must be a non-negative integer. If not, it is coerced to be zero or a positive
>integer.

>The variable x must be undefined before executing this function. Otherwise, the
>x in the returned expression for the polynomial will be automatically replaced
>with the value of x. Use `delVar x` if necessary before calling this function.

5.6   legendreqn(n)

Construct and return $Q_n(x)$, the Legendre function of the second kind.

Input arguments:

>n - designates which function to construct: $0, 1, \cdots, n$ specifies
>$Q_0(x), Q_1(x), \cdots, Q_n(x)$ respectively.

Returns:

A function in the symbolic variable x; *e.g.,* $\frac{1}{2}\ln(-\frac{x+1}{x-1})$.

Notes:

>n must be a non-negative integer. If not, it is coerced to be zero or a positive
>integer.

>The variable x must be undefined before executing this function. Otherwise, the
>x in the returned expression for the function will be automatically replaced with
>the value of x. Use `delVar x` if necessary before calling this function.

5.7 gamma(x)

Calculate and return the value of the Gamma function for the independent variable x.

Input arguments:

   x - The independent variable for the value $\Gamma(x)$.

Returns:

   The value $\Gamma(x)$ corresponding to the variable x.

Notes: none

5.8 lngamma(x)

Calculate and return the value of the natural logarithm of the Gamma function for the variable x.

Input arguments:

   x - The independent variable for the value $\ln \Gamma(x)$.

Returns:

   The value of the natural logarithm of $\Gamma(x)$.

Notes: Unlike some software implementations of the Log Gamma function, this implementation also returns $\ln \Gamma(x)$ for negative values of x.

6 Examples

Following are a few simple examples of problems that can be solved with the functions in the library.

6.1 Example 1:

Find the general solution of the differential equation

$$x^2 y'' + xy' + (x^2 - 9)y = 0$$

Solution: This equation is in the form of a Bessel equation with the $(x^2 - v^2)$ term as $(x^2 - 9)$. Solving for the x term and the v term implies the x term is $\sqrt{x^2} = x$ and the v term is $\sqrt{9} = 3$. The general solution is therefore

$$c_1 J_3(x) + c_2 Y_3(x)$$

6.2   Example 2:

Show that the general solution of the differential equation in example 1 is $c_1 J_3(x) + c_2 Y_3(x)$ when $x = 4.5$.

Solution: To show that $c_1 J_3(x) + c_2 Y_3(x)$ is the general solution to the differential equation, the values of y, y', and y'' for $J_3(4.5)$ and $Y_3(4.5)$ are needed. The values are then substituted in the differential equation to determine if the values result in a logical equality, $0 = 0$, which shows that the particular solutions are solutions to the equation. The general solution is then a linear combination of the particular solutions.

First, define the differential equation and extract its left-hand side:

$eqn1:=x^2 \cdot ypp+x \cdot yp+\left(x^2-9\right) \cdot y=0$ $x^2 \cdot \left(y+ypp\right)+yp \cdot x-9 \cdot y=0$

$lhs1:=\text{left}\left(eqn1\right)$ $x^2 \cdot \left(y+ypp\right)+yp \cdot x-9 \cdot y$

Execute the function `besselvalues()` to calculate the approximate values of $J_3(4.5)$ and its first and second derivatives. The values are returned in a three-element list. Substitute the values in the expression for the left-hand side of the equation to determine if the values satisfy the equation:

$j3vals:=despecfunc\backslash besselvalues\left(3,4.5,1\right)$

$\left\{0.424703972977,-0.065286998299,-0.22143842981\right\}$

$lhs1|x=4.5$ and $y=j3vals\left[1\right]$ and $yp=j3vals\left[2\right]$ and $ypp=j3vals\left[3\right]$ $^-1.\text{E}^-13$

Calculate the approximate values of $Y_3(4.5)$ and substitute the values in the expression for the left-hand side of the equation to determine if the values satisfy the equation:

$y3vals:=despecfunc\backslash besselvalues\left(3,4.5,2\right)$

$\left\{-0.009013681594,0.334490717723,-0.06932366972\right\}$

$lhs1|x=4.5$ and $y=y3vals\left[1\right]$ and $yp=y3vals\left[2\right]$ and $ypp=y3vals\left[3\right]$ $^-6.\text{E}^-14$

The values of the expression for the left-hand side of the equation after substituting the approximations are both close enough to zero to conclude that the particular solutions satisfy the equation. Since both particular solutions satisfy the equation, the linear combination of multiples of the particular solutions is the general solution of the equation.

Note: The calculated values are numerical approximations and are not exact. It is unlikely that the result of substituting the approximations in the expression for the left-hand side of the equation will be exactly zero.

The values of $J_3(x)$ and $Y_3(x)$ and their first and second derivatives can be viewed

in a graph page by selecting the **Document Tools - Trace - Trace All** menu item and positioning the 'trace' cursor at a value on the x axis. Figure 7 illustrates how the values for $J_3(x)$ and its first and second derivatives at $x = 4.5$ are displayed in a graph page.
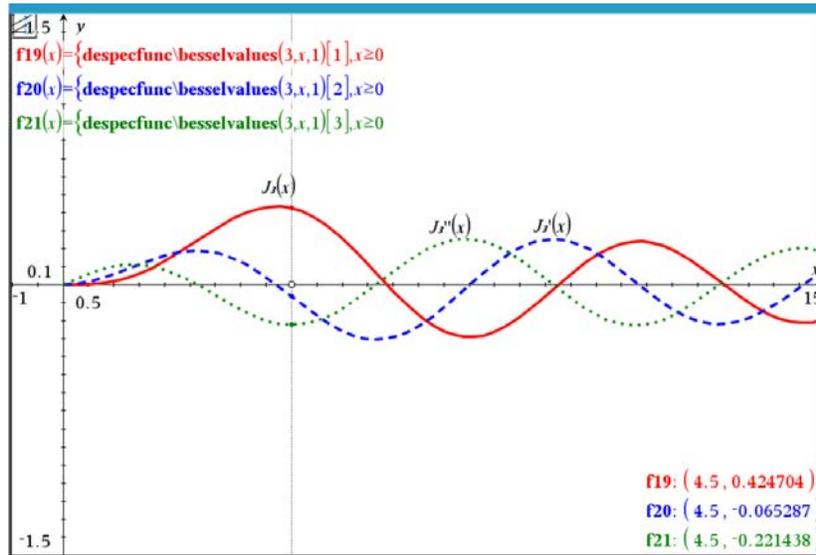

Figure 7: Values of $J_3(x)$ and Its Derivatives at $x = 4.5$

Figure 8 shows how the values for $Y_3(x)$ and its first two derivatives at $x = 4.5$ are displayed in a graph page.

6.3   Example 3:

Find the general solution of the differential equation

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

Solution: This equation is in the form of a Legendre equation with $n(n+1) = 6$, where n is the order of the Legendre functions that are solutions to the equation. Solving for n in a calculator page

$$\text{solve}\left(n \cdot (n+1) = 6, n\right) \qquad\qquad n = {}^-3 \text{ or } n = 2$$

yields $n = 2$ or $n = -3$. Since n must be ≥ 0, $n = 2$.

The general solution is
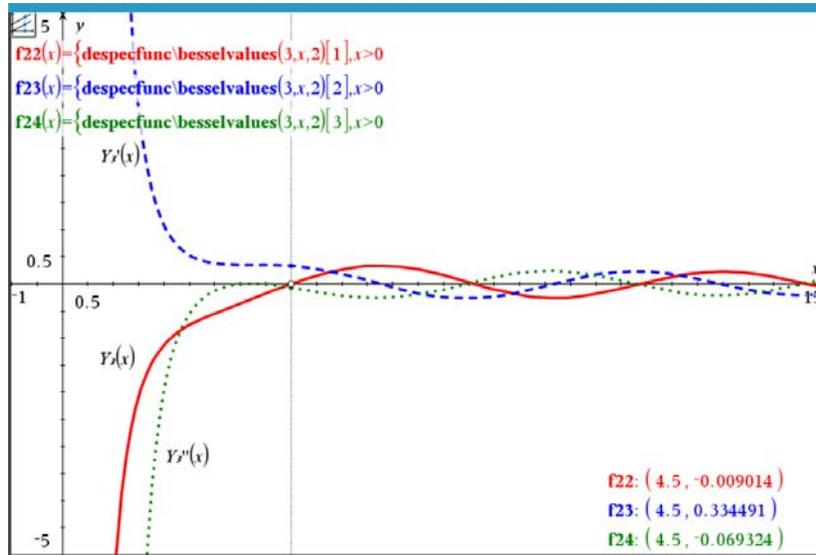$$c_1 P_2(x) + c_2 Q_2(x)$$

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Figure 8: Values of $Y_3(x)$ and Its Derivatives at $x = 4.5$

6.4   Example 4:

Find the values of $P_2(x)$ and $Q_2(x)$, the particular solutions of $(1 - x^2)y'' - 2xy' + 6y = 0$, and show that they are solutions to the differential equation.

Solution: The values are found by executing the functions `legendrepn(x)` and `legendreqn(x)` in a calculator page:

$$despecfunc\backslash legendrepn(2)$$

$$\frac{3 \cdot x^2 - 1}{2}$$

$$Done$$

$$p2(x) := \frac{3 \cdot x^2 - 1}{2}$$

$$despecfunc\backslash legendreqn(2)$$

$$\frac{\left(3 \cdot x^2 - 1\right) \cdot \ln\left(\frac{-(x+1)}{x-1}\right) - 6 \cdot x}{4}$$

$$Done$$

$$q2(x) := \frac{\left(3 \cdot x^2 - 1\right) \cdot \ln\left(\frac{-(x+1)}{x-1}\right) - 6 \cdot x}{4}$$

To show that these are solutions to the differential equation, find the first and second derivatives and plug the value of y and the derivatives into the equation.

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First, define the equation using yp for y' and ypp for y'':

© define the diff eqn

$diffeqn := (1-x^2) \cdot ypp - 2 \cdot x \cdot yp + 6 \cdot y = 0$ $-ypp \cdot x^2 - 2 \cdot yp \cdot x + 6 \cdot y + ypp = 0$

Next, substitute p2(x) for y, its first derivative for yp, and its second derivative for ypp in the equation using the constraint operator (|):

true

$diffeqn | y = p2(x)$ and $yp = \dfrac{d}{dx}(p2(x))$ and $ypp = \dfrac{d^2}{dx^2}(p2(x))$

The substitution results in a logical `true` value, meaning the substitution resulted in an equality: $0 = 0$. Therefore, the legendre polynomial of order 2 satisfies the differential equation and is a particular solution of the equation.

The same technique is used to show that q2(x) is a particular solution:

true

⚠ $diffeqn | y = q2(x)$ and $yp = \dfrac{d}{dx}(q2(x))$ and $ypp = \dfrac{d^2}{dx^2}(q2(x))$

The result of the substitution shows that q2(x) is a second particular solution of the differential equation.

Since multiples of solutions are also solutions, and linear combinations of solutions are solutions, the general solution to the Legendre equation is

$$c_1 P_2(x) + c_2 Q_2(x)$$

6.5   Example 5:

Find the values of the Gamma function and the Log-Gamma function for $x = \frac{5}{2}$ and $x = 4$, then convert the values of the Log-Gamma function to approximate values of the Gamma function.

Solution: The values using the library functions are:

$despecfunc \backslash gamma\left(\dfrac{5}{2}\right)$ 1.32934038818

$despecfunc \backslash lngamma\left(\dfrac{5}{2}\right)$ $\ln\left(\dfrac{625 \cdot \pi}{8}\right) - \dfrac{60707324}{24609375}$

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$despecfunc\backslash gamma(4)$ — 6.

$despecfunc\backslash lngamma(4)$ — $\dfrac{\ln(\pi)}{2}+\dfrac{15\cdot\ln(2)}{2}-\dfrac{328584643}{82575360}$

To convert the values of Log-Gamma to values of Gamma for positive values of x, simply exponentiate:

$e^{\text{approx}\left(despecfunc\backslash lngamma\left(\frac{5}{2}\right)\right)}$ — 1.32934016722

$e^{\text{approx}(despecfunc\backslash lngamma(4))}$ — 5.99999998303

The converted Log-Gamma values for x are approximately equal to the Gamma values for x.