Using the Differential Equations Series Library (deserieslib.tns)

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Typeset in LaTeX.

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Trademarks

1 Purpose

The functions in `deserieslib` generate series to approximate solutions to first- and second-order differential equations centered on either ordinary or regular singular points. The library can be installed and used with TI-Nspire CX CAS, TI-Nspire CAS Student Software, TI-Nspire CAS Teacher Software, and TI-Nspire CAS App for the iPad.

The functions in this library were implemented solely for finding series solutions for the types of differential equations commonly encountered in an introductory differential equations textbook. There is no guarantee that the functions can be used for more general differential equations.

2 Restrictions/Limitations

1. Complex solutions are not supported in `de_series_op()` or `de_series_sp()`.

2. Only homogeneous equations are supported.

3. Many Cauchy-Euler equations can not be solved with `de_series_sp()`. To solve these equations, use the function `de_indicial_eqn()` to obtain the values of the roots $r1$ and $r2$ and use those values to create closed-form solutions.

3 Library History

Date Created: 15June2018
Last Modified: 29September 2018 - Fixed bug in de_indicial_eqn()

4 Functions

The functions in the library and examples of using them are described in the following paragraphs. The examples shown below are screen-captures of commands and their results from TI-Nspire CAS Student Software.

4.1 help()

Displays usage information for the functions in the library.

4.1.1 Example

Select the help function in the library pane to add it to a calculator page, then execute the function:

```
deserieslib\help()
```

Scroll to display information about a library function:

```
3. de_singular_pts(equation,realonly)
   Input arguments:
   equation − the equation to examine to determine if there
      are singular points and their types.
   realonly − if true, return only real−valued singular points.
      if false, return both real and complex singular points.
   Returns:
   A matrix. Each row contains the value of the singular
      point in column1 and its type in column 2. Types
      are regular, irregular, removable.
----
```

4.2 de_series_op(eqn,order,op,showcoeffs)

Returns a series expanded around an ordinary point to approximate the solution of a first- or second-order differential equation.

Input arguments:

eqn - the differential equation
order - the order of the returned series
op - the ordinary point around which to expand the series
showcoeffs - display the 'raw' coefficients for the series

4.2.1 Example 1

This first example illustrates the general technique for using both `de_series_op()` and `de_series_sp()`.

To find a series solution for the DE $y'' - y = 0$ centered around the ordinary point $x = 0$, first define the equation, then call `de_series_op()` to return the series:

$de{:=}y''+y=0$	$y''+y=0$
$sol{:=}deserieslib\backslash de\_series\_op(de,5,0,\text{false})$	$\frac{a1\cdot x^5}{120}+\frac{a0\cdot x^4}{24}-\frac{a1\cdot x^3}{6}-\frac{a0\cdot x^2}{2}+a1\cdot x+a0$

The series contains $a0$ and $a1$, which are multipliers for the two independent solutions of the general solution. The two solutions can be extracted from the general solution using the constraint operator (|) to substitute values for $a0$ and $a1$:

$sol1{:=}sol\|a1=0$	$\frac{a0\cdot x^4}{24}-\frac{a0\cdot x^2}{2}+a0$
$sol2{:=}sol\|a0=0$	$\frac{a1\cdot x^5}{120}-\frac{a1\cdot x^3}{6}+a1\cdot x$

You may recognize these solutions as $a0$ times the taylor series for $\cos(x)$ and $a1$ times the taylor series for $\sin(x)$. This can be confirmed using the TI-Nspire CAS function `taylor()`:

$\text{expand}(a0\cdot \text{taylor}(\cos(x),x,5))$	$\frac{a0\cdot x^4}{24}-\frac{a0\cdot x^2}{2}+a0$
$\text{expand}(a1\cdot \text{taylor}(\sin(x),x,5))$	$\frac{a1\cdot x^5}{120}-\frac{a1\cdot x^3}{6}+a1\cdot x$

Setting the last argument to `de_series_op()` to `true` instructs the function to write the "raw" coefficients of the series:

$sol{:=}deserieslib\backslash de\_series\_op(de,5,0,\text{true})$

number of coefficients = **4** , coefficients:
$\{a3+20\cdot a5,a2+12\cdot a4,a1+6\cdot a3,a0+2\cdot a2\}$

$$\frac{a1\cdot x^5}{120}+\frac{a0\cdot x^4}{24}-\frac{a1\cdot x^3}{6}-\frac{a0\cdot x^2}{2}+a1\cdot x+a0$$

The raw coefficients can be used to manually solve for the coefficients of the series and derive a recurrence relation for the coefficients. In this example, neither $a0$ nor $a1$ are equal to zero, so $a2, a3, a4$, and $a5$ can be solved for in terms of $a1$ and $a2$: $a2 = -a0/2, a3 = -a1/6, \ldots.$

The coefficients in terms of $a1$ and $a2$ can also be found by copying the list of raw coefficients, assigning the list to a variable, setting each term in the list to 0, then using the TI-Nspire CAS function `linSolve()` to solve for the coefficients:

$sys := \{a3+20\cdot a5=0, a2+12\cdot a4=0, a1+6\cdot a3=0, a0+2\cdot a2=0\}$

$$\{a3+20\cdot a5=0, a2+12\cdot a4=0, a1+6\cdot a3=0, a0+2\cdot a2=0\}$$

$\text{linSolve}(sys, a2, a3, a4, a5)$

$$\left\{\frac{-a0}{2}, \frac{-a1}{6}, \frac{a0}{24}, \frac{a1}{120}\right\}$$

What if a solution around an ordinary point that is not equal to zero is wanted? The same technique as above is used, but the result from `de_series_op()` is a little different as shown in the next example.

4.2.2 Example 2

To find a series solution for the DE $y'' - xy = 0$ centered around the ordinary point $x = -2$, first define the equation, then call `de_series_op()` to return the series:

$de := y'' - x\cdot y = 0$ $\qquad y'' - x\cdot y = 0$

$xp := -2$ $\qquad -2$

$sols := deserieslib\backslash de\_series\_op(de, 5, xp, \text{false})$

$$\left\{\left(\frac{a1}{30}-\frac{a0}{15}\right)\cdot t^5+\left(\frac{a0}{6}+\frac{a1}{12}\right)\cdot t^4+\left(\frac{a0}{6}-\frac{a1}{3}\right)\cdot t^3-a0\cdot t^2+a1\cdot t+a0, \frac{-(2\cdot a0-a1)\cdot x^5}{30}-\frac{(6\cdot a0-5\cdot a1)\cdot x^4}{12}- \blacktriangleright\right.$$

When a series is not centered on $x = 0$, the function `de_series_op()` returns a list containing the solution in two forms: the first form is a series in terms of t, where $t = x - centerx$, and the second form is in terms of x, which is t expanded. Since the series for this example is centered on $x = -2$, $t = x - (-2) = x + 2$.

Note: the little arrow to the right of the returned value indicates the value is longer than the display area. To see the entire value, use the arrow keys to scroll.

The forms of the solution are retrieved from the list using list element indexing:

$solt := sols[1]$

$$\left(\frac{a1}{30}-\frac{a0}{15}\right)\cdot t^5+\left(\frac{a0}{6}+\frac{a1}{12}\right)\cdot t^4+\left(\frac{a0}{6}-\frac{a1}{3}\right)\cdot t^3-a0\cdot t^2+a1\cdot t+a0$$

$solx := sols[2]$

$$\frac{-(2\cdot a0-a1)\cdot x^5}{30}-\frac{(6\cdot a0-5\cdot a1)\cdot x^4}{12}-\frac{(7\cdot a0-10\cdot a1)\cdot x^3}{6}-\frac{4\cdot (a0-2\cdot a1)\cdot x^2}{3}-\frac{(6\cdot a0-7\cdot a1)\cdot x}{3}-\frac{17\cdot \blacktriangleright}{15}$$

The two independent solutions to the equation are obtained from the general solution by substituting $a0 = 1$ and $a1 = 0$ for one solution and $a0 = 0$ and $a1 = 1$ for the second solution. The two solutions in terms of t are:

solt1:=*solt*|*a0*=1 and *a1*=0	$\frac{-t^5}{15}+\frac{t^4}{6}+\frac{t^3}{6}-t^2+1$
solt2:=*solt*|*a0*=0 and *a1*=1	$\frac{t^5}{30}+\frac{t^4}{12}-\frac{t^3}{3}+t$

Note: replacing t with $(x+2)$ results in the solutions as presented in textbooks.

In terms of x, the two solutions are:

solx1:=*solx*|*a0*=1 and *a1*=0	$\frac{-x^5}{15}-\frac{x^4}{2}-\frac{7\cdot x^3}{6}-\frac{4\cdot x^2}{3}-2\cdot x-\frac{17}{15}$
solx2:=*solx*|*a0*=0 and *a1*=1	$\frac{x^5}{30}+\frac{5\cdot x^4}{12}+\frac{5\cdot x^3}{3}+\frac{8\cdot x^2}{3}+\frac{7\cdot x}{3}+\frac{26}{15}$

The solutions in terms of x are the solutions after replacing t with $(x+2)$ and expanding the solutions.

The general solution of the equation in terms of t is $a0(solt1)+a1(solt2)$ and in terms of x is $a0(solx1)+a1(solx2)$, which are the two forms of the solution returned by `de_series_op()`.

4.3 de_series_sp(eqn,order,sp,showcoeffs)

Returns a series expanded around a singular point to approximate the solution of a second-order differential equation.

Input arguments:

eqn - the differential equation
order - the order of the returned series
sp - the singular point around which to expand the series
showcoeffs - display the 'raw' coefficients for the series

This function returns an error message if

1. the differential equation is not a second-order equation,
2. the singular point is not a real-valued, regular singular point,
3. the roots of the indicial equation are imaginary,
4. the roots of the indicial equation are complicated fractions.

It is a good idea to use the functions `de_singular_pts()` and `de_indicial_eqn()` before using this function.

4.3.1 Example 3

Find a series solution for the DE $9x^2y'' + x^2y' + 2y = 0$ around the point $x = 0$.

It is apparent that since the coefficient of y'' equals 0 when $x = 0$, then 0 is a singular point. First, define the equation, then call the function `de_singular_pts()` to determine if 0 is a regular singular point:

$de{:=}9\cdot x^2\cdot y''{+}x^2\cdot y'{+}2\cdot y{=}0$	$x^2\cdot(9\cdot y''+y')+2\cdot y=0$
$deserieslib\backslash de\_singular\_pts(de,\text{false})$	$[0 \;\; \text{"regular"}]$

After verifying that $x = 0$ is a regular singular point, call the function `de_indicial_eqn()` to determine if the roots of the indicial equation are real integers or simple fractions:

$deserieslib\backslash de\_indicial\_eqn(de,0)$	$\left\{r^2-r+\frac{2}{9}=0,\frac{1}{3},\frac{2}{3}\right\}$

Finally, call `de_series_sp()` to find the two independent series solutions for the equation:

$sols{:=}deserieslib\backslash de\_series\_sp(de,3,0,\text{false})$

$$\left\{\frac{-x^{\frac{11}{3}}}{15309}+\frac{5\cdot x^{\frac{8}{3}}}{2268}-\frac{x^{\frac{5}{3}}}{18}+x^{\frac{2}{3}},\frac{-7\cdot x^{\frac{10}{3}}}{87480}+\frac{x^{\frac{7}{3}}}{405}-\frac{x^{\frac{4}{3}}}{18}+x^{\frac{1}{3}}\right\}$$

$sol1{:=}sols[1]$

$$\frac{-x^{\frac{11}{3}}}{15309}+\frac{5\cdot x^{\frac{8}{3}}}{2268}-\frac{x^{\frac{5}{3}}}{18}+x^{\frac{2}{3}}$$

$sol2{:=}sols[2]$

$$\frac{-7\cdot x^{\frac{10}{3}}}{87480}+\frac{x^{\frac{7}{3}}}{405}-\frac{x^{\frac{4}{3}}}{18}+x^{\frac{1}{3}}$$

The general solution is a linear combination of the two independent solutions: $gsol = C1 * sol1 + C2 * sol2$.

4.3.2 Example 4

Find a series solution for the DE $xy'' + y'' + y = 0$ around the point $x = 0$.

Following the same steps as in Example 3, define the equation, verify that 0 is a regular singular point and that the roots of the indicial equation are real integers or simple fractions:

$de{:=}x\cdot y''+y'+y=0$	$x\cdot y''+y'+y=0$
deserieslib\de_singular_pts(de,false)	$\begin{bmatrix}0 & \text{"regular"}\end{bmatrix}$
deserieslib\de_indicial_eqn(de,0)	$\{r^2=0,0\}$

Next, call `de_series_sp()` to find two independent solutions and extract the solutions from the returned list:

sols:=deserieslib\de_series_sp(de,5,0,false)

$$\left\{\frac{-x^5}{14400}+\frac{x^4}{576}-\frac{x^3}{36}+\frac{x^2}{4}-x+1, y1\cdot \ln(x)+\frac{137\cdot x^5}{432000}-\frac{25\cdot x^4}{3456}+\frac{11\cdot x^3}{108}-\frac{3\cdot x^2}{4}+2\cdot x\right\}$$

sol1:=sols[1]

$$\frac{-x^5}{14400}+\frac{x^4}{576}-\frac{x^3}{36}+\frac{x^2}{4}-x+1$$

sol2:=sols[2]|y1=sol1

$$\left(\frac{-x^5}{14400}+\frac{x^4}{576}-\frac{x^3}{36}+\frac{x^2}{4}-x+1\right)\cdot \ln(x)+\frac{137\cdot x^5}{432000}-\frac{25\cdot x^4}{3456}+\frac{11\cdot x^3}{108}-\frac{3\cdot x^2}{4}+2\cdot x$$

Notice that the returned second solution contains $y1$, which is the first solution. To retrieve the second solution, use the constraint operator to substitute the value of $sol1$ for $y1$. The general solution is a linear combination of the two independent solutions.

4.4 de_singular_pts(eqn,realonly)

Returns a list containing the singular points, if any, and their types, where `type` is `regular` or `irregular`. Returns an empty list if the equation does not have any singular points.

Input arguments:

eqn - the differential equation
realonly - if `true`, return only real-valued singular points

4.4.1 Example 5

Find any singular points and their types of the DE $(x^2-1)y''+y=0$:

$de1{:=}(x^2-1)\cdot y''+y=0$	$x^2\cdot y''-y''+y=0$
deserieslib\de_singular_pts(de1,false)	$\begin{bmatrix}-1 & \text{"regular"}\\ 1 & \text{"regular"}\end{bmatrix}$

4.4.2 Example 6

Find any singular points and their types of the DE $(x^2+1)y''+y=0$:

$de2:=(x^2+1)\cdot y''+y=0$	$x^2\cdot y''+y''+y=0$
$deserieslib\backslash de\_singular\_pts(de2,\text{false})$	$\begin{bmatrix} i & \text{"regular"} \\ -i & \text{"regular"} \end{bmatrix}$

4.4.3 Example 7

Find any singular points and their types of the DE $(x^2-x-2)^2y''+(x^2-4)y'=0$:

$de3:=(x^2-x-2)^2\cdot y''+(x^2-4)\cdot y'=0$	$x^4\cdot y''-2\cdot x^3\cdot y''+x^2\cdot (y'-3\cdot y'')+4\cdot x\cdot y''+4\cdot y''-4\cdot y'=0$
$deserieslib\backslash de\_singular\_pts(de3,\text{false})$	$\begin{bmatrix} -1 & \text{"irregular"} \\ 2 & \text{"regular"} \end{bmatrix}$

4.4.4 Example 8

Find any singular points and their types of the DE $(x^2-x-2)^2y''+(x^2-9)y'=0$:

$de4:=(x^2-x-2)^2\cdot y''+(x^2-9)\cdot y'=0$	$x^4\cdot y''-2\cdot x^3\cdot y''+x^2\cdot (y'-3\cdot y'')+4\cdot x\cdot y''+4\cdot y''-9\cdot y'=0$
$deserieslib\backslash de\_singular\_pts(de4,\text{false})$	$\begin{bmatrix} -1 & \text{"irregular"} \\ 2 & \text{"irregular"} \end{bmatrix}$

4.4.5 Example 9

Find any singular points and their types of the DE $y''+y=0$:

$de5:=y''+y=0$	$y''+y=0$
$deserieslib\backslash de\_singular\_pts(de5,\text{false})$	$\{\square\}$

4.5 de_indicial_eqn(eqn,sp)

Returns a list containing the indicial equation and its root or roots for a differential equation and a singular point. Returns an empty list and displays a warning message if the point `sp` is not a singular point or is not a regular singular point of the equation.

Input arguments:

eqn - the differential equation
sp - a regular singular point of the differential equation.

4.5.1 Example 10

Find the singular points and their indicial equation for the DE $9x^2y'' + 2y = 0$:

de1:=9· *x*^{2}· *y*''+2· *y*=0	9· *x*^{2}· *y*''+2· *y*=0
deserieslib\de_singular_pts(*de1*,false)	$\begin{bmatrix} 0 & \text{"regular"} \end{bmatrix}$
deserieslib\de_indicial_eqn(*de1*,0)	$\left\{ r^2-r+\frac{2}{9}=0, \frac{1}{3}, \frac{2}{3} \right\}$

4.5.2 Example 11

Find the singular points and their indicial equation for the DE $9x^2y'' + y = 0$:

de2:=9· *x*^{2}· *y*''+*y*=0	9· *x*^{2}· *y*''+*y*=0
deserieslib\de_singular_pts(*de2*,false)	$\begin{bmatrix} 0 & \text{"regular"} \end{bmatrix}$
deserieslib\de_indicial_eqn(*de2*,0)	$\left\{ r^2-r+\frac{1}{9}=0, \frac{-(\sqrt{5}-3)}{6}, \frac{\sqrt{5}+3}{6} \right\}$

4.5.3 Example 12

Find the singular points and their indicial equation for the DE $x^2y'' + x^2y' + y = 0$:

de3:=*x*^{2}· *y*''+*x*^{2}· *y*'+*y*=0	*x*^{2}· (*y*''+*y*')+*y*=0
deserieslib\de_singular_pts(*de3*,false)	$\begin{bmatrix} 0 & \text{"regular"} \end{bmatrix}$
deserieslib\de_indicial_eqn(*de3*,0)	$\left\{ r^2-r+1=0, \frac{1}{2}+\frac{\sqrt{3}}{2}\cdot \boldsymbol{i}, \frac{1}{2}-\frac{\sqrt{3}}{2}\cdot \boldsymbol{i} \right\}$

4.5.4 Example 13

Find the singular points and their indicial equation for the DE $(x^2 - x - 2)^2y'' + (x^2 - 4)y' = 0$:

$de4:=(x^2-x-2)^2 \cdot y''+(x^2-4)\cdot y'=0$ $\quad x^4\cdot y''-2\cdot x^3\cdot y''+x^2\cdot(y'-3\cdot y'')+4\cdot x\cdot y''+4\cdot y''-4\cdot y'=0$

$deserieslib\backslash de\_singular\_pts(de4,\text{false})$ $\quad \begin{bmatrix} -1 & \text{"irregular"} \\ 2 & \text{"regular"} \end{bmatrix}$

$deserieslib\backslash de\_indicial\_eqn(de4,-1)$

Warning: the value, -1, is not a regular singular point!

$\{\square\}$

$deserieslib\backslash de\_indicial\_eqn(de4,2)$ $\quad \left\{\frac{r\cdot(9\cdot r-5)}{9}=0,0,\frac{5}{9}\right\}$

$deserieslib\backslash de\_indicial\_eqn(de4,0)$

Warning: the value, 0, is not a singular point!

$\{\square\}$

4.6 bessel1(order,numterms)

Calculates a series for a bessel function of integer order.

Input arguments:

order - the integer order for the series
numterms - the number of terms for the series.

4.6.1 Example 14

Find 10 terms of the first four bessel functions of integer order, $Jn0, Jn1, Jn2$, and $Jn3$ and plot the four series:

$jn0:=deserieslib\backslash bessel1(0,10)$

$\frac{x^{20}}{13807847410237440000}-\frac{x^{18}}{34519618525593600}+\frac{x^{16}}{106542032486400}-\frac{x^{14}}{416179814400}+\frac{x}{21233}$ ▸

$jn1:=deserieslib\backslash bessel1(1,10)$

$x\cdot(x^{20}-440\cdot x^{18}+158400\cdot x^{16}-45619200\cdot x^{14}+10218700800\cdot x^{12}-1716741734400\cdot x^{10}+20$ ▸

$jn2:=deserieslib\backslash bessel1(2,10)$

$x^2\cdot(x^{20}-480\cdot x^{18}+190080\cdot x^{16}-60825600\cdot x^{14}+15328051200\cdot x^{12}-2942985830400\cdot x^{10}+4$ ▸

$jn3:=deserieslib\backslash bessel1(3,10)$

$x^3\cdot(x^{20}-520\cdot x^{18}+224640\cdot x^{16}-79073280\cdot x^{14}+22140518400\cdot x^{12}-4782351974400\cdot x^{10}+7$ ▸

The plot of the functions is